$$d^2F[\delta h_i, \delta h_j] = \int_{a=-\infty}^{b=\infty} \int_{-\infty}^{\infty} dt'' \, dt' (\alpha(t,t')\delta(t'-t'')) \times \delta h_i(t')\delta h_j(t'') \quad \text{(A3)}$$

From Eq. (A3) one can write the expression for the second functional derivative:

$$\frac{\delta^2 F[\delta h_i, \delta h_j]}{\delta h_i(t'')\delta h_j(t')} = \alpha(t,t')\delta(t'-t'') \quad \text{(A4)}$$

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P. Givi
Associate Editor

New Approach to Solution of the Falkner–Skan Equation

I. Sher* and A. Yakhot†
Ben-Gurion University of the Negev, Beer-Sheva 84105, Israel

Introduction

THE well-known general solution of laminar boundary-layer flows over two-dimensional wedges of the angle $\pi\beta$, discovered by Falkner and Skan in 1931, is appropriate for flows in which the freestream potential flow along the plate is given by $U(x) = cx^m$. Here, the exponent m, related to β by $m = \beta/(2-\beta)$, measures the pressure gradient in the stream direction. Introducing the nondimensional wall distance parameter η and transforming the stream function $\psi(x, y)$ lead to the similarity Falkner–Skan (FS) differential equation for the dependent variable $f(\eta)$:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \quad (1)$$

subject to the boundary conditions

$$f(0) = 0, \qquad f'(0) = 0, \qquad f'(\infty) = 1 \quad (2)$$

Here, the prime denotes differentiation with respect to the independent variable η defined by

$$\eta = y\left[\frac{(m+1)U(x)}{2\nu x}\right]^{\frac{1}{2}} \quad (3)$$

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*Graduate Student, Department of Mechanical Engineering.

†Associate Professor, The Pearlstone Center for Aeronautical Engineering Studies, Department of Mechanical Engineering.

and the similarity solution $f(\eta)$ is related to the stream function $\psi(x, y)$ by

$$\psi(x, y) = \left[\frac{2\nu U(x)}{m+1}\right]^{\frac{1}{2}} x^{\frac{1}{2}} f(\eta) \quad (4)$$

We recall that the velocity components u and v are expressed in terms of variables η and f as

$$u = Uf', \qquad v = -\left[\frac{\nu U(x)}{(2-\beta)x}\right]^{\frac{1}{2}} [f + (\beta - 1)\eta f'] \quad (5)$$

There is an enormous family of solutions to the FS equation. These solutions are of great significance for at least two reasons. First, in addition to flow along a flat plate, these give the flow near a forward stagnation point; second, they show the effects of pressure gradient on the velocity profile.

The essential difficulty in numerical solution of the FS equation stems from the two-point nature of its boundary conditions. For integration by the Runge–Kutta method, the third-order FS equation could be reduced to an equivalent system of three first-order equations convenient for numerical analysis, namely $f' = g$, $g' = h$, $h' = -fg - \beta(1 - g^2)$. The boundary conditions at $\eta = 0$ for f and $g = f'$ are known, namely: $f = g = 0$. Therefore, the problem is to find $h(0) = f''(0)$, which measures the shear friction at the wall, so that $f'(\infty) = 1$. This two-point boundary value problem poses a mathematical difficulty when one attempts to apply a simple shooting technique. With this method, the FS equation is integrated numerically from $\eta = 0$ with a guessed value of $f''(0)$. However, because of its nonlinear nature, numerical solutions diverge for all guessed values of $f''(0)$, except the "correct" guess. This means that solutions of the nonlinear FS equation exhibit a kind of spontaneous singularity as boundary conditions vary. These difficulties have been known for many years, and methods to overcome them have been discussed by many investigators in the past six decades. First, Hartree[1] studied the FS equation in detail. Falkner[2] gave further high-order accuracy solutions. Smith[3] and Evans[4] developed methods to determine a large enough value of $\eta_{\max}$, where $f''(0)$ is sought to yield $f''(\eta_{\max}) = 0$. Radbill[5] and Libby and Chen[6] have developed additional methods, which are quite useful for obtaining theoretical solutions in the range of $\beta < -0.19884$ (traditionally being the unstable separated flow region), as obtained by Libby and Liu.[7]

The purpose of this Note is to present a new numerical method for solving the FS boundary-layer equation in the range of $\beta \geq -0.19884$.

Numerical Procedure

We consider the FS equation at infinity (far from the wall). The asymptotic boundary conditions read: $f'(\eta \to \infty) \to 1$, $f''(\eta \to \infty) \to 0$. We develop a numerical procedure for integrating the FS equation from a given $\eta = \eta_{\max}$ downward to $\eta = \eta_0$, where $f'_0 = 0$. Because the FS equation does not contain η explicitly, we choose an arbitrary value of $\eta_{\max}$ and denote $f'(\eta_{\max}) = f'_{99}$, $f(\eta_{\max}) = f_{99}$. Here, f'_{99} is a certain specified value close to one, and f_{99} is finite. The correct choice of f''_{99}, which is close to zero, is more problematic and is the major difficulty for applying the boundary-layer-type equations at infinity. To compute and not to guess f''_{99}, we compute the expression $(f' - 1)/f''$ at $\eta = \eta_{\max}$. This expression is singular as $\eta_{\max} \to \infty$, and applying L'Hôpital's rule leads to

$$(f'_{99} - 1)/f''_{99} = f''_{99}/f'''_{99} \quad (6)$$

Substituting f'''_{99} from the FS equation leads to the quadratic equation for f''_{99}, which yields

$$f''_{99} = \tfrac{1}{2}(1 - f'_{99})\left[f_{99} + \sqrt{f_{99}^2 + 4\beta(1 + f'_{99})}\right] \quad (7)$$

This relation ensures that for an arbitrary f_{99} and for f'_{99} close to one, the asymptotic value of f''_{99} is computed using the original FS equation. For boundary-layer flows with no overshoot in the velocity profile (i.e., $f' < 1$), one has $[1 - f'(\eta \to \infty)] \to +0$, $f''(\eta \to \infty) \to +0$. Thus, the square root in the expression for f''_{99} is chosen with a positive sign.

Table 1 Comparison of exact and calculated values of f_0''

		$f_{99}'=0.90$		$f_{99}'=0.95$		$f_{99}'=0.99$	
β	Exact f_0''	f_0''	Error %	f_0''	Error %	f_0''	Error %
−0.18	−0.09769[a]	−0.04726	51.622	−0.08251	15.539	−0.09584	1.894
	0.12864	0.05361	58.326	0.10348	19.558	0.12552	2.425
0	0.46960	0.45236	3.671	0.46274	1.461	0.46869	0.194
0.3	0.77476	0.76625	1.098	0.77139	0.435	0.77426	0.065
1	1.23259	1.22861	0.323	1.23150	0.088	1.23245	0.011
2	1.68722	1.68481	0.143	1.68651	0.042	1.68716	0.004
10	3.67523	3.67401	0.033	3.67507	0.004	3.67546	0.006

[a]This value is not presented in Ref. 9 and has been calculated using the standard shooting method.

Finally, we formulate a Cauchy problem for the FS equation, which could be solved by the Runge–Kutta method:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{8}$$

$$f(\eta_{\max}) = f_{99}, \qquad f'(\eta_{\max}) = f_{99}'$$

$$f''(\eta_{\max}) = f_{99}'' = \tfrac{1}{2}(1 - f_{99}')\left[f_{99} + \sqrt{f_{99}^2 + 4\beta(1 + f_{99}')}\right] \tag{9}$$

To solve the original FS problem, we suggest the following numerical procedure: 1) for a given $\eta_{\max}$, guess f_{99} and specify f_{99}' close to one; 2) integrate the FS equation downward in the $-\eta$ direction to the point $\eta = \eta_0$, where $f_0' = 0$. The value of the solution obtained at this point, f_0, is the corrector to the f_{99} guess to obey the impermeable wall-boundary condition, $f_0 = 0$.

Results and Discussion

The suggested method was applied to calculate f_0'' for different values of β in the range $\beta \geq -0.19884$. For $\beta = -0.19884$, the solution describes the separation limit[8] when $f_0'' = 0$. To derive the numerical procedure, we employed L'Hôpital's rule to finite values of f_{99}', which introduces the only source of numerical error. Computations have been performed with a double-precision accuracy for three values of $f_{99}' = 0.90, 0.95, 0.99$ to illustrate its influence on the method's accuracy. The results are summarized in Table 1. For comparison, the values of f_0'', mentioned in the literature, are taken from White[9] and referred to as "exact" values. It is obvious that as f_{99}' approaches one, the computed value of f_0'' becomes more accurate, as can be readily seen from Table 1. Our numerical results show that the choice of $f_{99}' = 0.99$ yields remarkably accurate values of f_0'' in the entire range of wedge angles β considered. It is worth noting that the error is smaller for higher values of β. This is attributed to the lower sensitivity of the equation as β increases. (The flow is more stable as the favorable pressure gradient is larger.)

For $\beta \geq 0$, the solutions to the FS equation are known to be unique, without backflow regions.[10] Therefore, only one point $\eta = \eta_0$ of zero longitudinal velocity at the wall ($f_0' = 0$) exists in the interval $\eta_0 \leq \eta \leq \eta_{\max}$. As a result, $f'' > 0$ in this interval. In the range $-0.19884 < \beta < 0$, two physically acceptable solutions exist: without and with backflow regions.[8] The solutions that include backflow must have two points of zero longitudinal velocity (one of them at the wall). As a result, two points of $f' = 0$ exist in the interval $\eta_0 \leq \eta \leq \eta_{\max}$. The first is a local minimum point, where $f'' > 0$, and the second one is a local maximum point, where $f'' < 0$. If the first point of $f' = 0$ is considered as the last point of integration (the wall), the backflow-free solution is obtained. To obtain solutions that include backflow, the second point of $f' = 0$ should be considered as the last point of integration. These solutions include two points of $f' = 0$, and the backflow exists between them. For $\beta = -0.19884$ (separation limit, where $f_0'' = 0$), the solution is known to be unique; therefore, only one point of $f' = 0$ exists down the η scale, which is a deflection point ($f'' = 0$).

A certain class of similarity solutions for wedges with permeable walls can also be obtained using the suggested method. As it follows from Eq. (5), a nonzero (positive or negative) value of the corrector at the wall, f_0, can be considered as a boundary condition parameter of suction or blowing, respectively. For every f_0 above the physical limit (where blowoff of the boundary layer occurs), a physically acceptable solution exists. It means that for such values of f_0, corresponding values of f_{99} exist, as shown in Fig. 1. The minimum physical value of f_0 corresponds to the minimum physical f_{99}. As a consequence only values of f_{99}, which are above the physical minimum, yield a point $\eta = \eta_0$ where $f' = 0$. Therefore, the first guess for f_{99} in the shooting procedure should be large enough.

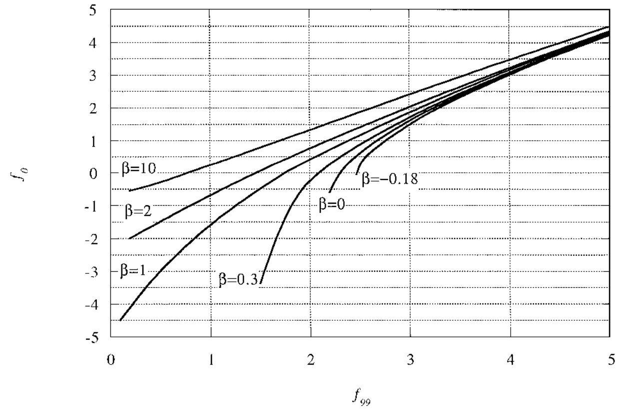

Fig. 1 Values of f_0 vs f_{99} for $f_{99}' = 0.99$.

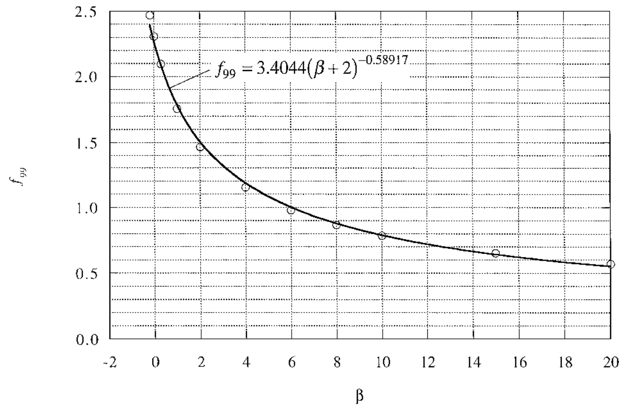

Fig. 2 Values of f_{99} vs β for $f_{99}' = 0.99$. Impermeable wall conditions ($f_0 = 0$).

For $f_{99}' = 0.99$ and impermeable wall conditions ($f_0 = 0$), the calculated f_{99} for different β are presented in Fig. 2. Using these results, a simple correlation between f_{99} and β has been derived: $f_{99} = 3.4044(\beta + 2)^{-0.58917}$.

Conclusions

A new numerical method for solving the FS equation has been developed. The method is applicable for permeable/impermeable wall conditions in the range of $\beta \geq -0.19884$. For the negative β range, two possible solutions are obtainable. The method suggests integrating the FS equation downward in the $-\eta$ direction from the $\eta = \eta_{\max}$ location, where the value of $f''(\eta_{\max})$ is computed to ensure the solution's convergence.

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M. Sichel
Associate Editor

Evolution Strategies for Automatic Optimization of Jet Mixing

Petros Koumoutsakos*
Swiss Federal Institute of Technology, CH-8092 Zurich, Switzerland
Jonathan Freund†
University of California, Los Angeles, Los Angeles, California 90095-1597
and
David Parekh‡
Georgia Tech Research Institute, Atlanta, Georgia 30080

I. Introduction

EVOLUTION strategies (ES) are introduced for the optimization of active control parameters for enhancing jet mixing. It is shown that the evolution algorithms can identify, in an automated fashion, not only previously known effective actuations but also find good but previously unidentified parameters. In this study, simulations of model jets are used to demonstrate the feasibility of the methods. ES are robust, highly parallel, and portable algorithms that may be most useful in an experimental setting at realistic Reynolds numbers. Simulations of inviscid incompressible flows using vortex models, as well as direct numerical simulations (DNS) of very low-Reynolds-number compressible flows, are used in this study to evaluate different forcing parameters.

Our objective is twofold: 1) explore the possibility of ES to find previously identified modes of efficient operation and 2) discover previously unknown effective actuations. Practical engineering concerns will dictate the choice of actuator parameters and relevant cost functions. In Sec. II, we present a description of the ES; in Sec. III, we present results from the application of these to the optimization of compressible jets and vortex models. Section IV is a discussion of results and an outline for future research.

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*Professor, Institute of Computational Sciences; petros@inf.ethz.ch.

†Assistant Professor, Department of Mechanical and Aerospace Engineering; jfreund@ucla.edu. Member AIAA.

‡Director, Aerospace, Transportation, and Advanced Systems Laboratory, Georgia Tech Research Center; david.parekh@gtri.gatech.edu. Member AIAA.

II. Evolution Strategies

ES are continuous parameter optimization techniques based on principles of evolution such as reproduction, mutation, and selection. We define a vector in the control parameter space $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ as an individual and a set of such individuals as a population. ES use a fitness value, prescribed by $F(\boldsymbol{x}) = F(x_1, x_2, \ldots, x_M)$, to identify the best individual from a population. We take better individuals to have larger F values.

A. Two-Membered ES

The simplest ES has a population with two competing individuals, a two-membered strategy. Evolution occurs by mutation and selection, the two operations that Darwin considered the most important in the evolution of species. Each individual is represented by a pair of real vectors $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{\sigma})$, where $\boldsymbol{\sigma}$ is an M-dimensional vector of standard deviations.

Following Rechenberg,[1] the optimization algorithm is as follows:

1) Initialization is where a parent genotype consisting of M genes is specified initially as $\boldsymbol{x}^0$.

2) Mutation is when the parent of generation n produces a descendant with slightly different genotype. The operation of mutation is realized by modifying $\boldsymbol{x}_p$ according to $\boldsymbol{x}_c^n = \boldsymbol{x}_p^n + \mathcal{N}(\boldsymbol{\sigma}_p^n)$, where $\mathcal{N}(\boldsymbol{\sigma})$ is an M-dimensional vector of normal random number with zero mean and standard deviations $\boldsymbol{\sigma}$.

3) Selection is where the fittest individual according to its F value becomes the parent of the next generation:

$$\boldsymbol{x}_p^{n+1} = \begin{cases} \boldsymbol{x}_p^n, & \text{if } F\left(\boldsymbol{x}_p^n\right) \geq F\left(\boldsymbol{x}_c^n\right) \\ \boldsymbol{x}_c^n, & \text{otherwise} \end{cases} \tag{1}$$

The variance of the population members is adjusted using the one-fifth success rule proposed by Rechenberg: If more than one in five offsprings result in an improved solution, then the variance is increased.[1] For regular optimization problems[2] the method is known to converge to a global minimum, but the rate of convergence cannot be anticipated. Therefore, in finite time there is, of course, no guarantee that the global optimum has been reached, a trait shared by all optimization techniques. Schwefel[3] provides a complete description of the algorithm.

B. Parameter Constraints

In problems of active flow control, engineering considerations impose constraints on the actuation parameters. Such constraints are formulated as inequalities, such as $C_j(\boldsymbol{x}) \geq 0$. In this work descendants that do not satisfy the constraints are treated as unsuccessful mutations.

III. Jet Flows Optimization

A. Optimized Excitation of Compressible Jets

DNS of the developing region of compressible jets forced by slot-jet fluidic actuators are used to evaluate the fitness of individuals. The compressible flow equations were solved directly using a combination of sixth-order compact finite differences, spectral methods, and fourth-order Runge–Kutta time advancement. Further details of the numerical algorithm and techniques for including two slot-jet actuators that each span 90 deg of the jet (on opposite sides) just downstream of the nozzle were reported elsewhere.[4] Despite the low Reynolds number dictated by the computational expense (only $Re = 500$ in this study) the actuators were able to induce the gross effects observed in experiments at much higher Reynolds numbers.[5] A path to implementation at more relevant flow conditions is discussed in Sec. IV.

For this study, only three actuation parameters were varied: amplitude, frequency, and phase. The actuation is taken as a sum of harmonic waveforms:

$$v_r = -\min\left(\sum_{i=1}^{N} A_i\left[1 + \sin\left(U_j\frac{Sr_i}{D}t + \phi_i\right)\operatorname{sgn}[\cos\theta]\right], \frac{U_j}{2}\right) \tag{2}$$

where v_r is the radial velocity at the actuator exit, U_j is the jet exit velocity, A_i are the amplitudes, Sr_i are the Strouhal numbers, and